Ranking Systems

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Ranking systems intro

- In social theory a set of agents/voters are called to rank set of alternatives;
- Given individual preference, social ranking of alternatives is generated;
- Theory studies desired properties (PE, IIA, ...) of aggregation of agents' ranking into a social ranking;
- Page ranking is a special setting of social ranking where set of agents and set of alternatives coincide;

PageRank

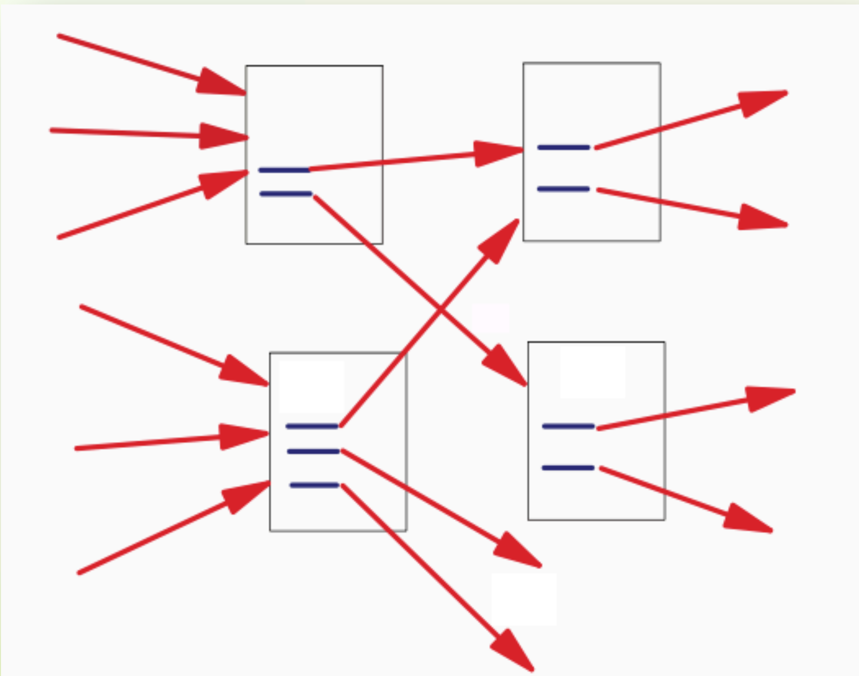
- Made Google exist (and flourish);
- Set of agents vote for each other by having URL links (point to each other);
- Note: this complies with our early definition of the “game”
  - Set of agents: pages themselves;
  - Set of actions: “link to”;
  - Outcome (payoffs): relative ordering of pages;
- Setting is slightly different
  - Credibility of votes

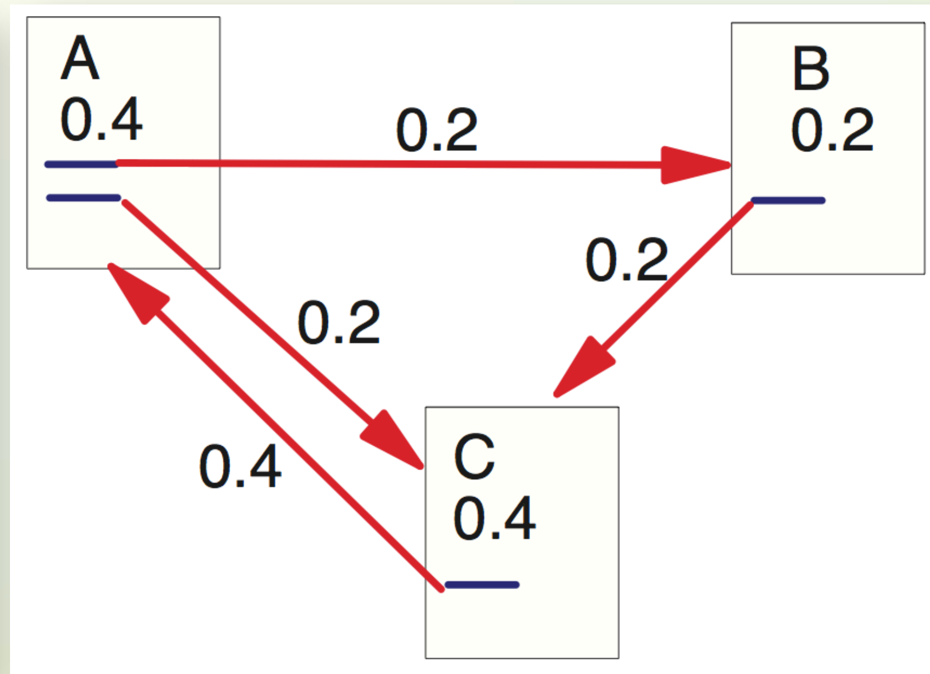
Fun Game

- Free dinner ticket for Green College tonight is limited (say it is only N/3);
- Kevin wants to be fair and give it to those whom "society" ranks highest;
- Vote for 1 or 2 "deserving" people in the class
  - Criteria: friend, working hard, look hungry and etc.
  - Please do not vote for yourself ☺
- In a given paper, put your name and max 2 other names, whom you want to award the ticket.

Contents

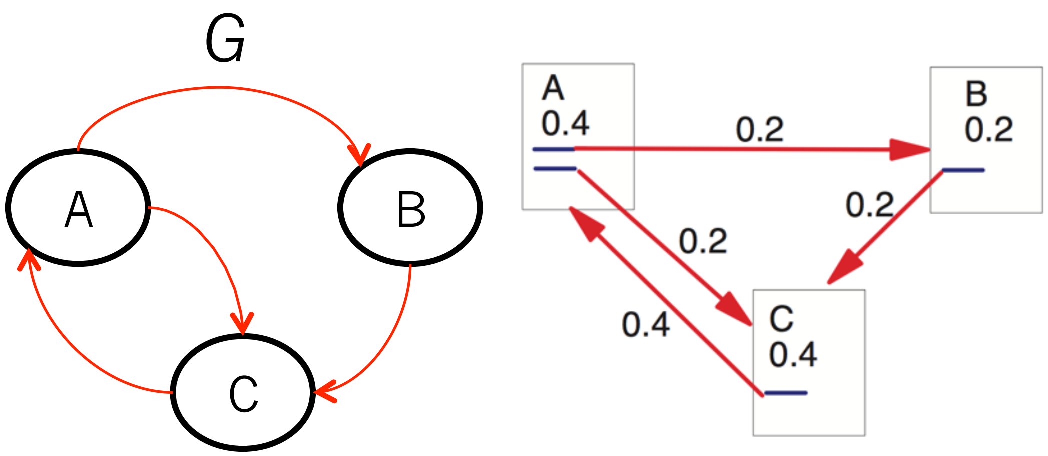
- Introduction to Ranking System
- Fun game
- PageRank: bringing order to the Web
  - Algorithmic (computational) perspective
- Representation theorem for PageRank
  - Axioms
  - Properties
- “PageRank” coincidence
- Conclusion and recap




A
0.4
0.2
B
0.2
0.2
0.2
0.4
C
0.4

MyRank vs. PageRank

- MyRank ranks vertices in G in ascending order of the number of incoming links.



MyRank(G): A=B<C

PageRank(G): A=C>B

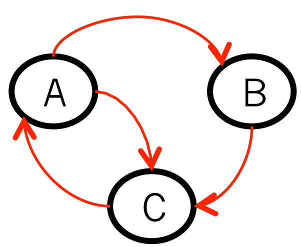
Define PageRank

Definition 2.4. Let $G = (V, E)$ be a directed graph, and let $v \in V$ be a vertex in G. Then: The *successor set* of v is $S_G(v) = \{u | (v, u) \in E\}$, and the *predecessor set* of v is $P_G(v) = \{u | (u, v) \in E\}$.

Definition 2.5. Let $G = (V, E)$ be a directed graph, and assume $V = \{v_1, v_2, \dots, v_n\}$. the *PageRank Matrix* A_G (of dimension $n \times n$) is defined as:

$$[A_G]_{i,j} = \begin{cases} 1/|S_G(v_j)| & (v_j, v_i) \in E \\ 0 & \text{Otherwise.} \end{cases}$$

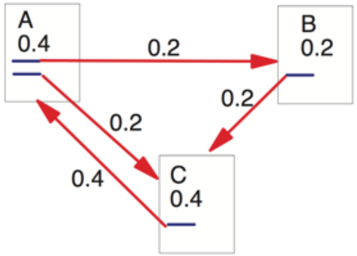
G



A_G

	a	b	c
a	0	0	0.4
b	0.2	0	0
c	0.2	0.2	0

converged



Definition 2.6. Let $G = (V, E)$ be some strongly connected graph, and assume $V = \{v_1, v_2, \dots, v_n\}$. Let $\mathbf{r}$ be the unique solution of the system $A_G \cdot \mathbf{r} = \mathbf{r}$ where $r_1 = 1$. The *PageRank* $PR_G(v_i)$ of a vertex $v_i \in V$ is defined as $PR_G(v_i) = r_i$. The *PageRank ranking system* is a ranking system that for the vertex set V maps G to $\preceq_G^{PR}$, where $\preceq_G^{PR}$ is defined as: for all $v_i, v_j \in V$: $v_i \preceq_G^{PR} v_j$ if and only if $PR_G(v_i) \leq PR_G(v_j)$.

Are definitions "interesting"?

- Pros
  - Defines powerful heuristics for the ranking of Internet pages;
  - Adopted "as-is" by Google's search engine;
  - Computationally efficient;
- Cons
  - Numeric procedure;
  - Does not really talk about "ranking system properties"
- Recall Arrow's powerful and beautiful axiomatic theorem

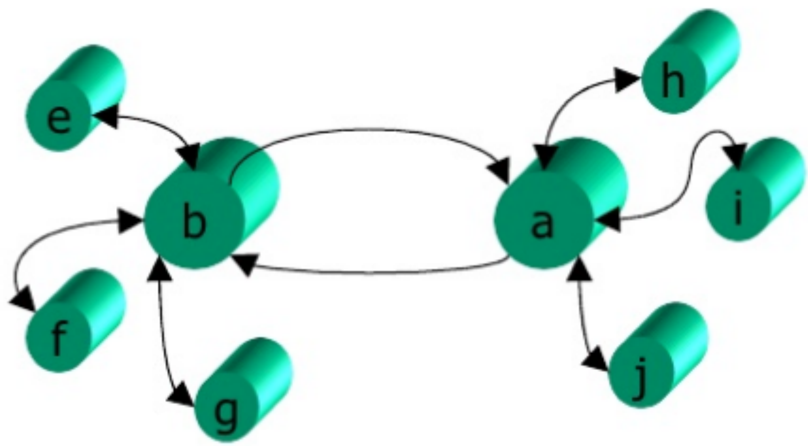
Theorem 9.4.4 (Arrow, 1951) *If $|O| \geq 3$, any social welfare function W that is Pareto efficient and independent of irrelevant alternatives is dictatorial.*

- Can we come up with axioms for ranking system?

Contents

Axioms – Isomorphism

Axiom 3.1. *(Isomorphism) A ranking system F satisfies* isomorphism *if for every isomorphism function* $\varphi : V_1 \mapsto V_2$, *and two isomorphic graphs* $G \in \mathbb{G}_{V_1}, \varphi(G) \in \mathbb{G}_{V_2}$: $\preceq^F_{\varphi(G)} = \varphi(\preceq^F_G)$.



e = f = g = h = i = j

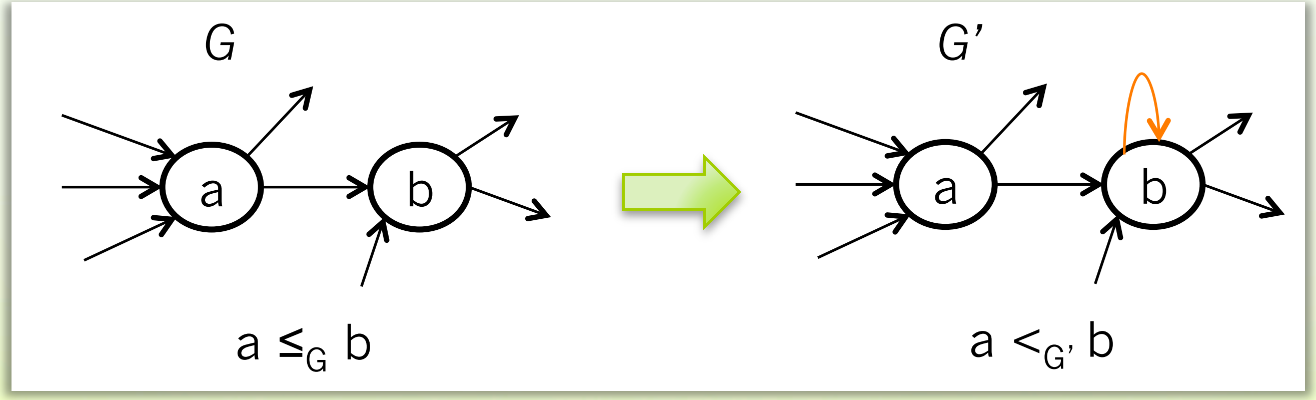
a = b

Intuition: Ranking is independent of vertex names;
Consequence: symmetric vertices have the same rank.

Axioms – Self Edge

Notation: Let $G = (V, E) \in \mathbb{G}_V$ be a graph s.t. $(v, v) \notin E$. Let $G' = (V, E \cup \{(v, v)\})$. Let us denote $\mathbf{SelfEdge}(G, v) = G'$ and $\mathbf{SelfEdge}^{-1}(G', v) = G$. Note that $\mathbf{SelfEdge}^{-1}(G', v)$ is well defined.

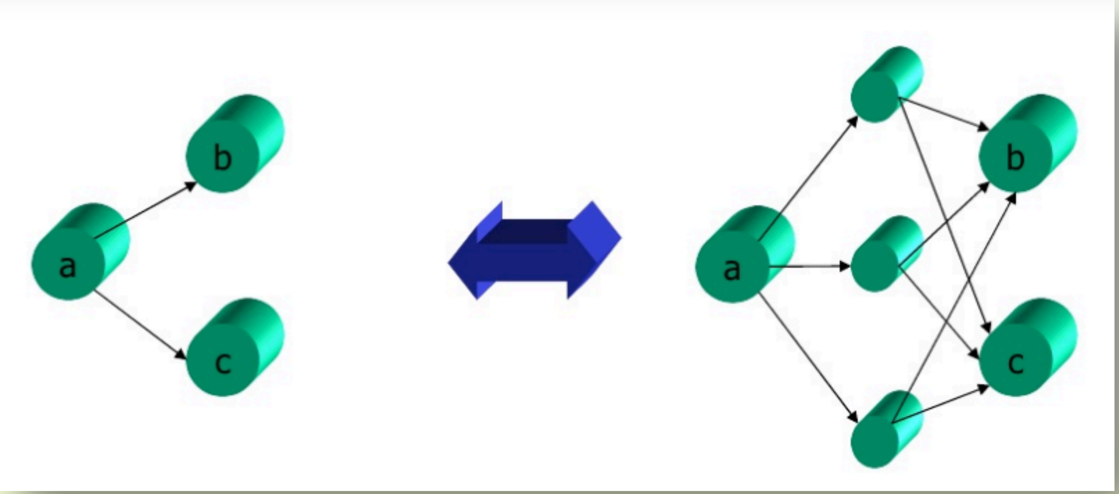
Axiom 3.2. *(Self edge) Let F be a ranking system. F satisfies the* self edge *axiom if for every vertex set V and for every vertex $v \in V$ and for every graph $G = (V, E) \in \mathbb{G}_V$ s.t. $(v, v) \notin E$, and for every $v_1, v_2 \in V \setminus \{v\}$: Let $G' = \mathbf{SelfEdge}(G, v)$. If $v_1 \preceq^F_G v$ then $v \not\preceq^F_{G'} v_1$; and $v_1 \preceq^F_G v_2$ iff $v_1 \preceq^F_{G'} v_2$.*



Intuition: Page can increase its rank by linking to itself, but relative ranking of everything else remains unchanged.

Axioms – Vote by Committee

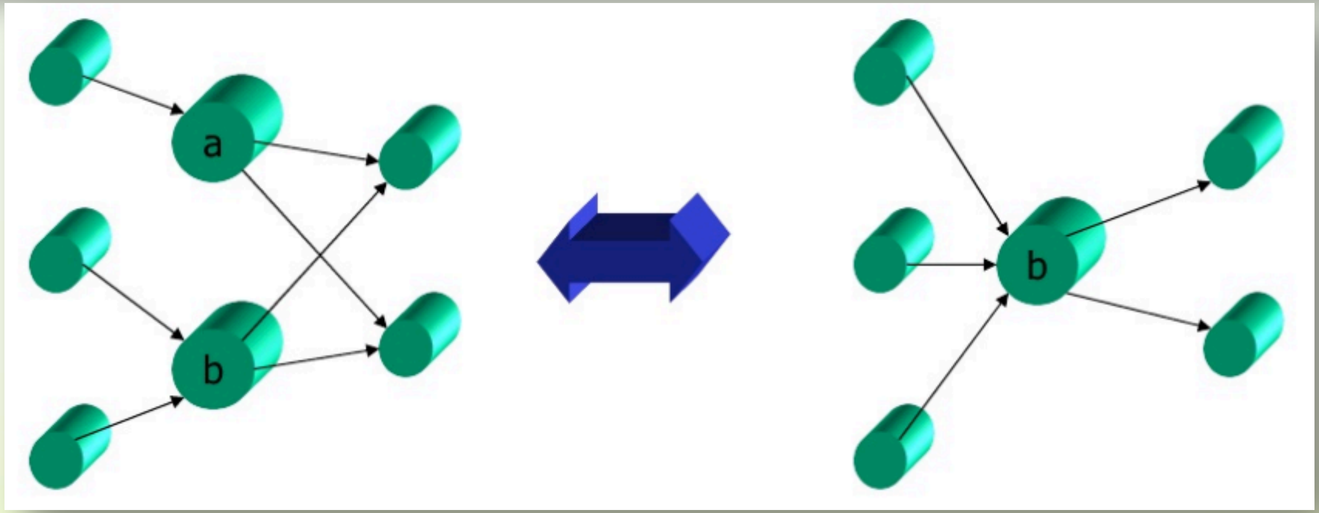
Axiom 3.3. *(Vote by committee) Let F be a ranking system. F satisfies* vote by committee *if for every vertex set V, for every vertex $v \in V$, for every graph $G = (V, E) \in \mathbb{G}_V$, for every $v_1, v_2 \in V$, and for every $m \in \mathbb{N}$: Let $G' = (V \cup \{u_1, u_2, \ldots, u_m\}, E \setminus \{(v, x) | x \in S_G(v)\} \cup \{(v, u_i) | i = 1, \ldots, m\} \cup \{(u_i, x) | x \in S_G(v), i = 1, \ldots, m\})$, where $\{u_1, u_2, \ldots, u_m\} \cap V = \emptyset$. Then, $v_1 \preceq_G^F v_2$ iff $v_1 \preceq_{G'}^F v_2$.*



Intuition: The importance **a** is providing for **b** and **c** should not change due to the fact that **a** assigns his power through committee.

Axioms – Collapsing

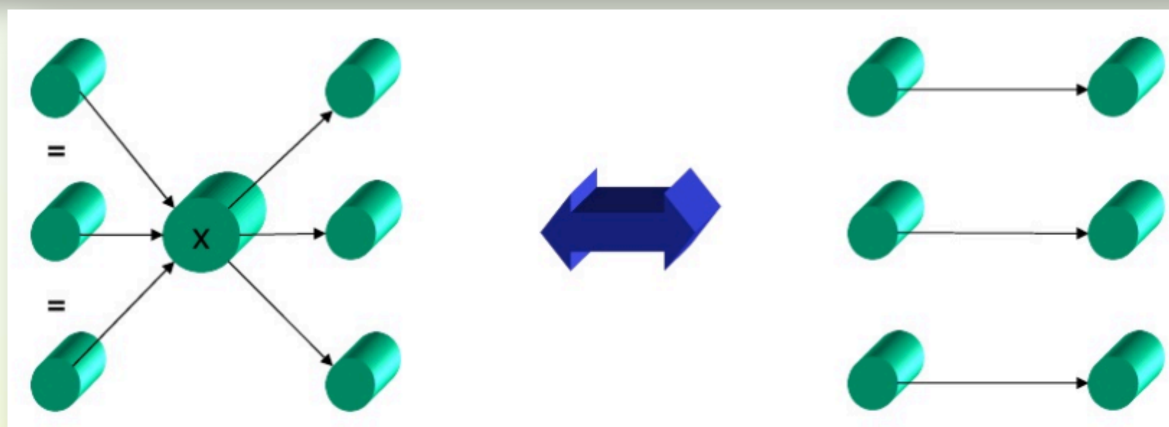
Axiom 3.4. *(collapsing) Let F be a ranking system. F satisfies* collapsing *if for every vertex set V, for every $v, v' \in V$, for every $v_1, v_2 \in V \setminus \{v, v'\}$, and for every graph $G = (V, E) \in \mathbb{G}_V$ for which $S_G(v) = S_G(v')$, $P_G(v) \cap P_G(v') = \emptyset$, and $[P_G(v) \cup P_G(v')] \cap \{v, v'\} = \emptyset$: Let $G' = (V \setminus \{v'\}, E \setminus \{(v', x) | x \in S_G(v')\} \setminus \{(x, v') | x \in P_G(v')\} \cup \{(x, v) | x \in P_G(v')\})$. Then, $v_1 \preceq_G^F v_2$ iff $v_1 \preceq_{G'}^F v_2$.*



Intuition: If **a** and **b** gets vote from disjoint agents and their successors coincide, collapse of **a** to **b** should not change relative ordering of other pages. Neither **a** nor **b** had the self edge.

Axioms – Proxy

Axiom 3.5. *(proxy) Let F be a ranking system. F satisfies* proxy *if for every vertex set V, for every vertex $v \in V$, for every $v_1, v_2 \in V \setminus \{v\}$, and for every graph $G = (V, E) \in \mathbb{G}_V$ for which $|P_G(v)| = |S_G(v)|$, for all $p \in P_G(v)$: $S_G(p) = \{v\}$, and for all $p, p' \in P_G(v)$: $p \simeq_G^F p'$: Assume $P_G(v) = \{p_1, p_2, \ldots, p_m\}$ and $S_G(v) = \{s_1, s_2, \ldots, s_m\}$. Let $G' = (V \setminus \{v\}, E \setminus \{(x, v), (v, x) | x \in V\} \cup \{(p_i, s_i) | i \in \{1, \ldots, m\}\})$. Then, $v_1 \preceq_G^F v_2$ iff $v_1 \preceq_{G'}^F v_2$.*



Intuition: pages that link to **x** could pass directly the importance to pages that **x** link to, without using **x** as a proxy for distribution.

PageRank with axioms

Proposition 3.6. *The PageRank ranking system PR satisfies isomorphism, self edge, vote by committee, collapsing, and proxy.*

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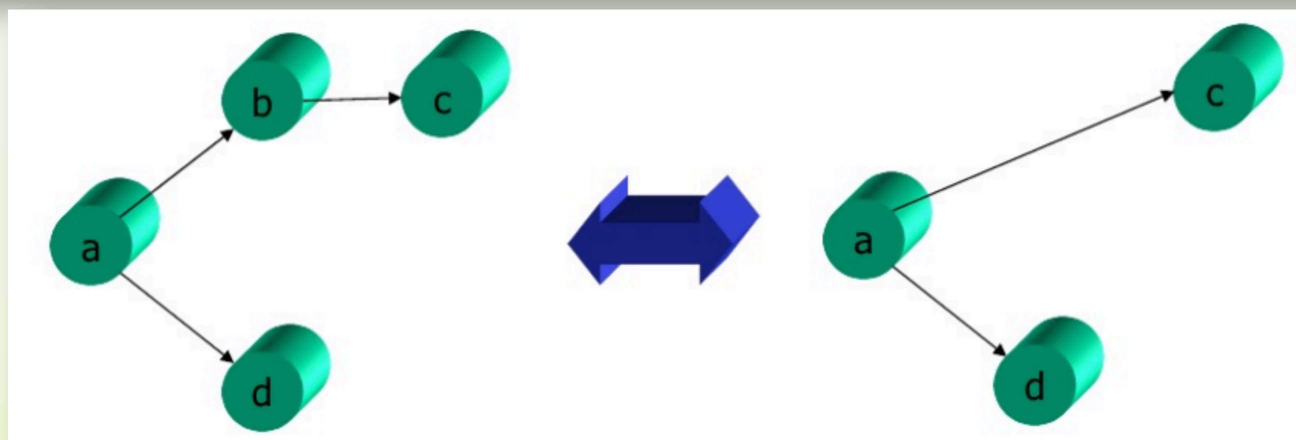
Properties – Del(.,.)

Definition 4.1. Let F be a ranking system. F has the *weak deletion* property if for every vertex set V, for every vertex $v \in V$ and for all vertices $v_1, v_2 \in V \setminus \{v\}$, and for every graph $G = (V, E) \in \mathbb{G}_V$ s.t. $S(v) = \{s\}$, $P(v) = \{p\}$, and $(s, p) \notin E$: Let $G' = \mathbf{Del}(G, v)$. Then, $v_1 \preceq^F_G v_2$ iff $v_1 \preceq^F_{G'} v_2$.

Lemma 4.2. *Let F be a ranking system that satisfies isomorphism, vote by committee and proxy. Then, F has the weak deletion property.*

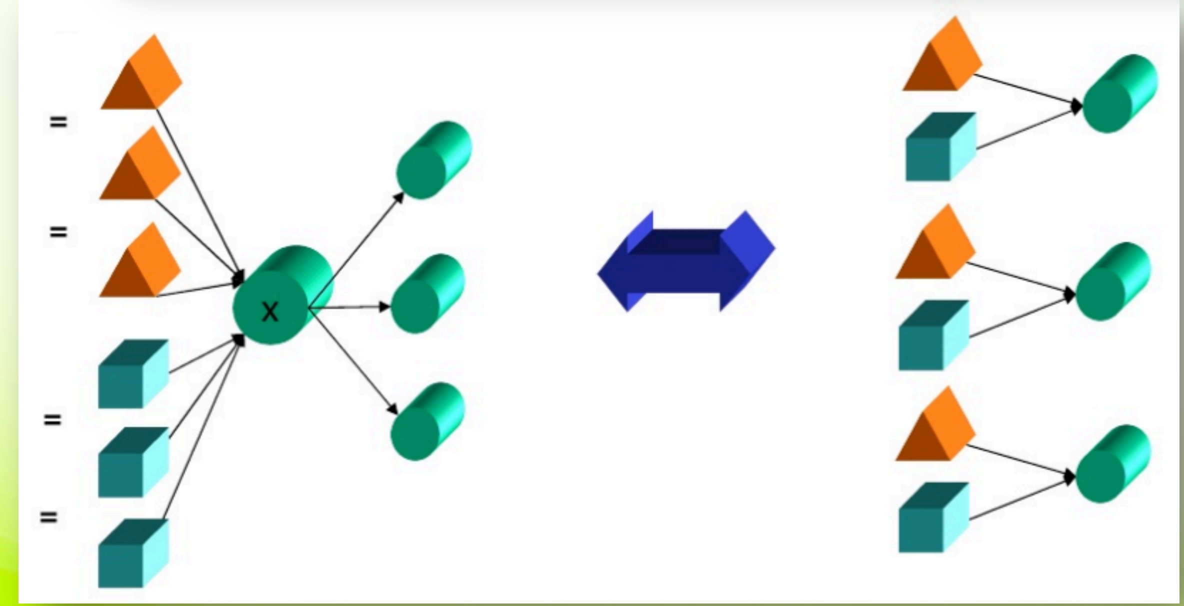


Notes: |P(b)| = |S(b)|=1;
There is no direct edge between **a** and **c**.

Properties – Delete(.,.,.)

Definition 4.3. Let F be a ranking system. F has the *strong deletion* property if for every vertex set V, for every vertex $v \in V$, for all $v_1, v_2 \in V \setminus \{v\}$, and for every graph $G = (V, E) \in \mathbb{G}_V$ s.t. $S(v) = \{s_1, s_2, \ldots, s_t\}$, $P(v) = \{p_j^i | j = 1, \ldots, t; i = 0, \ldots, m\}$, $S(p_j^i) = \{v\}$ for all $j \in \{1, \ldots t\}$ and $i \in \{0, \ldots, m\}$, and $p_j^i \simeq_G^F p_k^i$ for all $i \in \{0, \ldots, m\}$ and $j, k \in \{1, \ldots t\}$: Let $G' = \mathbf{Delete}(G, v, \{(s_1, \{p_1^i | i = 0, \ldots m\}), \ldots (s_t, \{p_t^i | i = 0, \ldots m\})\})$. Then, $v_1 \preceq_G^F v_2$ iff $v_1 \preceq_{G'}^F v_2$.

Lemma 4.4. *Let F be a ranking system that satisfies collapsing and proxy. Then, F has the strong deletion property.*

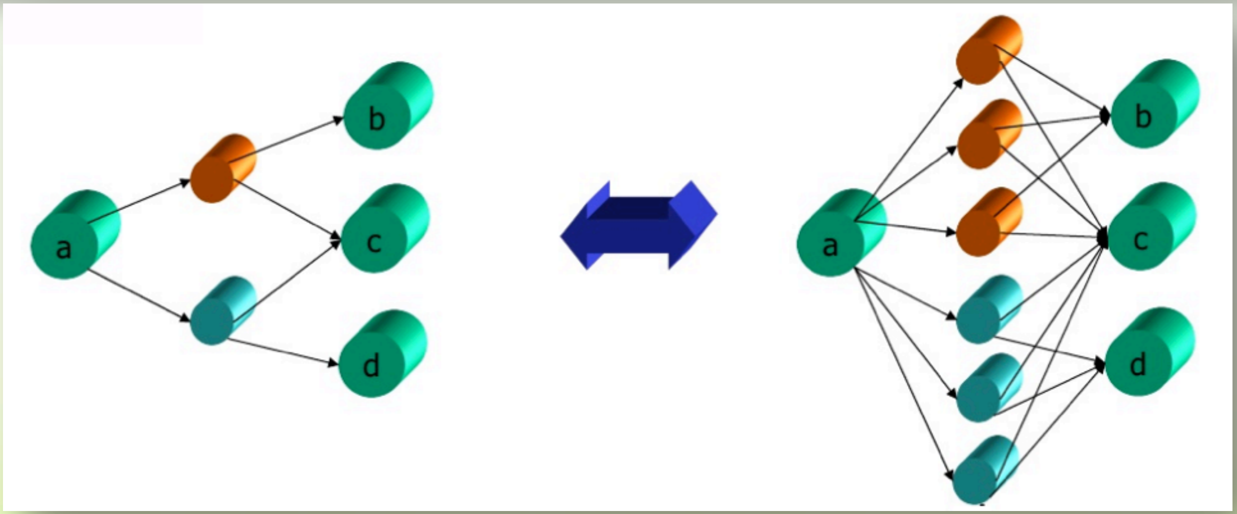


Notes:
- Nodes in P(x) have no other outgoing edges.
- x has no other edges.

Properties – Duplicate(.,.,.)

Definition 4.5. Let F be a ranking system. F has the *edge duplication* property if for every vertex set V, for all vertices $v, v_1, v_2 \in V$, for every $m \in \mathbb{N}$, and for every graph $G = (V, E) \in \mathbb{G}_V$: Let $S(v) = \{s_1^0, s_2^0, \ldots, s_t^0\}$, and let $G' = \mathbf{Duplicate}(G, v, m)$. Then, $v_1 \preceq_G^F v_2$ iff $v_1 \preceq_{G'}^F v_2$.

Lemma 4.6. *Let F be a ranking system that satisfies isomorphism, vote by committee, collapsing, and proxy. Then, F has the edge duplication property.*



Notes: All successors of **a** duplicated the same number of times. There are no edges from **S(a)** to **S(a)**.

PageRank coincidence

Proposition 3.6. *The PageRank ranking system PR satisfies isomorphism, self edge, vote by committee, collapsing, and proxy.*

Theorem 5.1. *A ranking system F satisfies isomorphism, self edge, vote by committee, collapsing, and proxy if and only if F is the PageRank ranking system.*

Proposition 5.2. *Let F_1 and F_2 be a ranking systems that have the weak deletion, strong deletion, and edge duplication properties, and satisfy the self edge and isomorphism axioms. Then, F_1 and F_2 are the same ranking system (notation: $F_1 \equiv F_2$).*

Conclusion

- Connects algorithms and Internet technologies to the mathematical theory of social choice;
- Sets axiomatic foundation to ranking systems
  - opens venue to define other ranking systems axiomatically and evaluate (perhaps compare) their properties;
  - difficult (if not impossible) to do in algorithmic (computational) representation;

Recap

- Introduction to ranking systems
  - Special setting of social choice, agents and alternatives coincide
- PageRank
  - Computation and axiomatic
  - Properties which axioms guarantee
- PageRank coincidence

Thank you!

References

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5. Trust among strangers in internet transactions: Empirical analysis of eBay' s reputation system, by Paul Resnick, Richard Zeckhauser, The Economics of the Internet and E-commerce (Advances in Applied Microeconomics, Volume 11)
6. Original slides from reference (1) http://www.slideshare.net/Adaoviedo/ranking-systems-4505613

Define Ranking System

Definition 2.1. A directed graph $G = (V, E)$ is called *strongly connected* if for all vertices $v_1, v_2 \in V$ there exists a path from v_1 to v_2 in E.

Definition 2.2. Let A be some set. A relation $R \subseteq A \times A$ is called an *ordering on* A if it is reflexive, transitive, complete and anti-symmetric. Let $L(A)$ denote the set of orderings on A.

Notation: Let $\preceq$ be an ordering, then $\simeq$ is the equality predicate of $\preceq$. Formally, $a \simeq b$ if and only if $a \preceq b$ and $b \preceq a$.

Definition 2.3. Let $\mathbb{G}_V$ be the set of all strongly connected graphs with vertex set V. A *ranking system* F is a functional that for every finite vertex set V maps every strongly connected graph $G \in \mathbb{G}_V$ to an ordering $\preceq_G^F \in L(V)$.

PageRank satisfies SelfEdge

Axiom 2: Self Edge (SE)

- Node v has a self-edge (v,v) in G′, but does not in G. Otherwise G and G′ are identical. F satisfies SE iff for all $u,w \neq v$:

  $(u \leq v \rightarrow u <' v)$ and $(u \leq w \Leftrightarrow u \leq' w)$

- PageRank satisfies SE:
  Suppose v has k outgoing edges in G. Let $(r_1, \ldots, r_v, \ldots, r_N)$ be the rank vector of G, then $(r_1, \ldots, r_v+1/k, \ldots, r_N)$ is the rank vector of G′

Proof sketch

- Define SCDG **G=(V,E)** and **a**,**b** in **V**;
- Eliminate all other nodes in **G** while preserving the relative ranking of **a** and **b**;
- In the resulting graph **G'** the relative ranking of **a** and **b** given by the axioms can be uniquely determined;
- Therefore the axioms rank any SCDG uniquely.
- It follows that all ranking systems satisfying the axioms coincide.